
APPLIED MECHANICS – Critical and historical examination of the principal theories or solutions concerning the equilibrium of arches; by M. Poncelet.

A translation, by Mike Clarke, of an article by J-V Poncelet which appeared in Comptes rendus hebdomadaires des séances de l'Académie des sciences, tome 35 (juillet-décembre 1852), pp494-502 and subsequent issues

Jean-Victor Poncelet on Skew Arches (1852) – translated from the original by Mike Clarke

The following is a translation of an article by J-V Poncelet which appeared in **Comptes rendus hebdomadaires des séances de l'Académie des sciences**, tome 35 (juillet-décembre 1852), pp 494-502 and subsequent issues

The original French is available at <<https://gallica.bnf.fr/ark:/12148/bpt6k2992n/f498.item>>

APPLIED MECHANICS – *Critical and historical examination of the principal theories or solutions concerning the equilibrium of arches*; by M. Poncelet.

Questions relating to the stability of buildings, those which concern in particular the equilibrium of the vaults, join to a high practical utility a theoretical importance which one cannot ignore according to the great number of attempts which, until now, were made by the geometers to subject the data to the calculation. These questions have acquired, by the immense works of the railways in progress or in project, such an interest of topicality, that the author of this Notice, already so extensive and yet very incomplete, thought to render a true service to the engineers, by putting before their eyes the summarized table of the principal researches concerning the theory of vaults, and which were able to exert, directly or indirectly, a certain influence on the solutions today admitted in practice. The extreme divergence of opinions relative to the real usefulness of such research, the very obscurity and uncertainties that one notices in the numerous works written on this thorny subject, have, on the other hand, imposed upon us the duty of not approaching the Academy of Sciences with the analysis of the important Memoirs presented, in the last place, by Messrs Yvon Villarceau and J. Carvallo, without having made a comparative and sufficiently thorough examination of the research of their predecessors, limiting ourselves, however, to those of them which have acquired some value in the eyes of persons who do not disdain to be guided by the lights of theory.

Before Coulomb, we only had mathematical considerations on the equilibrium of vaults, and very imperfect rules of thumb, based on limited hypotheses, and most of them lacking the precision and certainty which alone can recommend them to the confidence of enlightened engineers. Parent, Couplet, Belidor and Bossut, in France; Gregory, Whewell, Emerson, Hutton, in England; Lorgna, Mascheroni, etc., and before them, Lahire (*Mémoires de l'Académie des Sciences*, 1712), by neglecting all influence of cohesion and friction on the planes of the joints, had considered the upper part of the vault as a kind of wedge acting symmetrically, on either side of the key, to overturn the lateral and lower parts by rotation around the external edge of the base of the straight legs; for it had been felt from the outset that the hypothesis of the polishing of the joints was in no way admissible for these latter parts.

We will not insist here on the first researches by which one hoped, in this same hypothesis of the corner without friction or cohesion, to determine the form of greater stability of a vault, according to various conditions or hypotheses, in particular the form of the extrados, when that of the intrados is given a priori, which, by considering the equilibrium of each voussoir in isolation, led to infinite thicknesses or widths of the joints towards the origin of the vault, contrary to the day-to-day indications of the experiment. Nevertheless, this solution, in which the mutual pressures of the segments are supposed to be normal to the joint planes, had the advantage that it made builders feel, early on, the importance of extra-dimensioning the vaults in such a way as to progressively increase their thicknesses, going from the top to the base. Even today, when the influence of friction on the conditions of stability of vaults is well recognized and subjected to calculation, according to the beautiful discoveries of Coulomb, the research of their most advantageous form the stability, or the most economic, still preserves, as one will see it, its practical importance when one pretends to take into account the true elements of the question.

On the other hand, the determination of the thickness of the footings of the vaults, according to Lahire's method, although based on the hypothesis, also precarious and deduced from some vague data of experience, that the joints of rupture which limit on both sides the upper part, acting by

sliding like a wedge, divide symmetrically each of the half-vaults into equal parts. This determination, which gives thrusts that are generally too strong, and, consequently, thicknesses of the footings that are very likely to ensure the necessary excess stability, has nevertheless served as a basis for the establishment of useful Tables drawn up by the famous engineers Perronet and de Chezy; for it is necessary, whatever one does, that the most eminently practical rules, those, for example, which served the architects of the Middle Ages for the construction of our beautiful Gothic churches, and which have been transmitted to us by Delarue, Frezier, etc., were originally derived from some theoretical conceptions, more or less rational or empirical. The Tables of Perronet, which were, until recently, used by the Ponts et Chaussées for the establishment of bridge arches, were established by making some corrections to the formula of Lahire for the case of arches with a basket-handle design. In the case of the lowered basket-handle arches, whose breaking joints were chosen at 60° above the horizontal of the origin (naissances), at the same time as the calculation data relating to the piers and abutments of the large openings were reduced by one foot to one and a half; Corrections which, without any doubt, have as a starting point the very observations of l'abbé Perronet, continued by other famous engineers, de Prony, Gauthey, Rondelet, etc., on the phenomena of subsidence which are observed during the stopping of deformation (decintrement) of the vaults of large bridges.²

In his *Memoir* of 1773, published among those of the foreign scholars of the Academy, under the title of Application of the rules of maximis and minimis to some problems of statics relating to architecture, Coulomb, to whom one cannot deny the quality of practitioner as having belonged to the corps of the military Engineers, indicated the first, in a precise way, the true conditions of the balance and the stability of the barrel vaults, supposed symmetrical with regard to the vertical plane which divides the key in equal parts. After indicating and demonstrating the insufficiency of the old solutions, where friction on the joint planes was neglected, and giving a luminous outline of the questions which relate to the form of equilibrium of the vaults in this hypothesis, Coulomb brings in the consideration of this friction and of the cohesion; from which results, not simply the possibility of the sliding of the voussoirs one on the other, but also of their rotation around the extreme edges of the joints, in the regions where the rupture can be virtually made. He thus determines, for each case of equilibrium or distinct mode of failure, the position of the joints where sliding and rotation are most likely to occur, as well as the corresponding lower and upper limits of the horizontal thrust or resultant of the pressures exerted at the key, the intensity of which is remarkable in that it is absolutely independent of the height of the straight legs. A passage in this *Memoir* warns that most of the cases of rupture of equilibrium refer, contrary to Lahire's hypotheses, to the rotation and division of the vault into four portions, two by two symmetrical, rotating around their extreme edges. This passage gives reason to think that Coulomb had been aware of the results of experiments carried out in 1732 by Danisy, of the Academy of Montpellier, on models of vaults established on a scale too small to lead to precise results (*Traité de la Coupe des pierres*, by Frézier, volume III). But, in recommending especially to the engineers the solution relating to this last case of equilibrium, Coulomb is careful to note that the resultant of the pressures on the joints of rupture must deviate enough from the edge of rotation of these joints, to avoid the crushing of the neighbouring part, and he recalls, on this subject, the theoretical rules which, In another part of his Memoir, he established the theoretical rules relating to the maximum resistance of masonry pillars, rules which nevertheless seem to be difficult to apply to the case of vaults where the real extent of the support surface and the distribution of pressures on the joints remain undetermined in the condition of true stability.

The generality and vagueness in which Coulomb had left himself on this point and that which concerns the position of the thrust at the key, the very lack of example or of any application of the principles to special cases, are sufficient to explain how the beautiful and useful conceptions of this

¹ Sganzin, *Cours de constructions* (1809)

² *Mémoires de Académie des Sciences* de 1773; Nouvelle architecture hydraulique de Prony; Traité de la construction des ponts par Gauthey; Art de bâtir, par Rondelet.

illustrious engineer had remained, until recent times, in a complete oblivion, in spite of their scientific and practical value.

The old experiments of Gauthey and Rondelet, those of the chief engineer of the Ponts et Chaussées, Boistard, carried out in 1800, on models of a fairly large size, and where suitable precautions had been taken to bring to light the true laws of the phenomenon of the rupture of vaults by rotation, The theoretical tests which followed and which led to considering the vaults as likely to break into four parts rotating, by their extreme edges, around each other, like so many articulated levers, these experiments and these tests did a great service to the engineers, by making them completely abandon the old theories of Lahire and Belidor, in order to return to a mode of solution less marred by arbitrariness.

But this new way of considering the equilibrium of vaults was, in its turn, too exclusive, in that no account was taken of the possibility of sliding;³ it also offered complications of which Coulomb's method was free, and which arose principally from the supposition that the parts inferior to the joint of rupture of the kidneys can exert an influence on the determination of the maximum horizontal thrust at the keystone. As it also, this theory did not indicate any precise means of regulating the extra thickness to be given to the right-hand feet of the vault, to ensure them an essential additional stability, in conformity with that which the already existing constructions have, and that the practical Tables hitherto accepted in the Ponts et Chaussées also indicated - a thickening of which Gauthey mainly attributed the necessity to the defects inherent in the constitution of the ground of the foundations, and to the chances of crushing of the external edge of rotation of the right feet, the resultant of the efforts tending to be transferred.

Navier, in reproducing the ideas of Gauthey and Boistard in the Notes with which he enriched the new edition of Belidor's *Science des Ingénieurs*, added useful developments and remarks, especially from the point of view of the pressures supported by the voussoirs and the arches. But we owe more particularly to M. Audoy, then chief of battalion of the Engineers, for having, the first, brought the engineers back to the general and luminous methods of Coulomb. In developing and rectifying the incomplete theories mentioned above, in his beautiful Memoir inserted in the fourth number of the *Mémorial de l'Officier du Génie* (year 1820), he pointed out that they, as well as the facts deduced from the results of the experiment, fell within the consequences which could have been immediately concluded from the theory of this illustrious physicist. The analytical formulas established by M. Audoy, as a necessary continuation of these same doctrines, for round arches, circular arches, basket-handle arches, and variously extradosed arches, have rendered the greatest services to engineers, and have since formed the basis of the teaching of artillery and engineering students at the Metz School of Application.

In these formulas, it is assumed that the thickness of the vault at the key was determined by the practical rule adopted by Perronet, or according to other equivalent data, that in a word, the shape and dimensions of this vault were fixed according to the nature of the construction and the data of the experiment; then the joints of rupture of the kidneys which correspond to the maximum of the horizontal thrust at the key in the hypothesis of the strict equilibrium by sliding or rotation of the upper parts, from which one deduces then the thickness to be given to the straight legs of the vault, are sought analytically. But, as any construction of this kind requires an excess of stability to avoid the various accidents which can result either from the defect of the foundations, or from accidental overloads, from malfunctions or from any kind of collapse, or even because of the compressibility of the materials and the soil, M. Audoy investigates, in the example of constructions already old and still existing, what is the coefficient or numerical multiplier which it is necessary to apply to the value of the thrust, provided by the conditions of strict equilibrium, to give to each kind of vault the additional stability indicated by experience; and this method, already adopted by M. Francais for the coverings destined to support the thrust of the earth, has thus brought the question out of the

³ It had not been observed in the small-scale experiments referred to, and could not be observed under the conditions in which it was placed; but its effects on the equilibrium of vaults were demonstrated in special experiments by G. Atwood, published in 1801, in which the voussoirs were made of polished copper at the joints.

unfortunate state of uncertainty in which it had hitherto remained, by banishing from it, so to speak, all empiricism or arbitrariness.

The coefficient of stability which is at issue, and which basically amounts to increasing in a certain proportion the moment of resistance or of the straight legs, this coefficient, quite close to the number 2, for the most solid buildings, such as the large bridges and the powder magazines of the Vauban system, seems to be sufficient, in fact, to reassure against the chances of accidents, when the vault is established according to the rules of the art and in sufficiently resistant materials; because, by its adoption, the resultant of the thrust at the top and the weight of the lower parts, comes to meet the base of the right foot, at a distance from the external edge of rotation, always higher than the quarter of the effective width of this base, for the lightest vaults or the most elevated right feet, and which increases as the vault is stronger, more overloaded at the top or receives a larger opening,⁴ to such an extent that, for the vaults of large bridges and gunpowder magazines, it notably exceeds half the thickness of the right foot.

The learned engineer to whom we owe this progress in the question of vaults, considered in its applications to the most commonly used constructions, did not fail to show how the principles put forward by Coulomb could also be used to ascertain the state of solidity and stability in the various other parts of the vault, and in particular at the right of the naissances. Unfortunately, the extreme and inevitable complication of the formulas to which it is arrived, and which were to replace the method of purely geometrical approximation indicated by this illustrious physicist, the length even of the calculations or gropings necessary to determine numerically, in each case, the value of the maximum of thrust and the position of the joint of rupture which depend on the resolution of a transcendent equation, these difficulties have engaged, later, various military engineers to concern themselves with geometrical or analytical methods capable of simplifying the applications of the formulae to practice, simplifications which the author had called for towards the end of his important *Memoir*, and which have moreover made it possible to extend and deepen, more and more, the various questions which relate to the stability of vaults considered from this same point of view.

Before coming to these simplifications of Mr. Audoy's formulas, we will mention, in the first place, the remarkable *Memorandum on the stability of vaults*, written, in Russia, by Messrs. Lamé and Clapeyron, on the occasion of the reconstruction of the church of Saint-Isaac in Saint-Petersburg, and which was, in May 1823, the subject of a very favourable *Report* by M. de Prony, made to the Academy of Sciences, in his name and in that of M. Ch. Dupin (*Annales des Mines*, tome VIII, year 1823). MM. Lamé and Clapeyron adopting exclusively the hypothesis of the rupture by rotation of cylindrical vaults, in the form of four levers articulated at both ends and without slippage considering, moreover, the stability of such a vault or of any portion of vault, as measured by the excess of the moment of resistance over that of power, an excess whose value remains arbitrary here, they are led, for the determination of the joints of rupture or of maximum thrust, to results

⁴ It is easy to see, in fact, geometrically, that the fictitious resultant of this thrust, thus increased, and of the weight of the lower parts being subject, in the calculations, to pass through the outer edge of the base of the right foot, whose position it serves to determine, it necessarily happens that the effective resultant, the one corresponding to the simple thrust, deduced from the conditions of strict equilibrium and of the maximum, divides, approximately in equal parts, the interval comprised between the edge of rotation of which it is a question and the lowered vertical of the centre of gravity of the whole mass of the half-arch and the right foot. Now this vertical, always obviously beyond the axis of the latter in relation to this same edge, for the thinnest vaults and the most elevated right feet, moves away from it progressively as the thickness and the opening of the vault increase, with equal height and overload of the right foot. It is also noted that, for the low heights of the latter, for the births in particular, and when the vault, reduced to a simple strip, is newly off-centre, the vertical of the general centre of gravity falling much within the base of support or of the bearing, the coefficient of stability 2 would give the resultant of which it was spoken, a direction which would itself go beyond the middle of this base; This is more harmful than useful, and shows that the coefficient per rotation must then approach unity, as the example of existing constructions also shows. It would seem appropriate, in these circumstances where the consideration of rebates and overloads is neglected, to prefer to use the stability coefficient, the rule which consists of determining the thickness at the beginning of the vaults according to the condition that the resultant of the forces passes through the middle of the support base, or at least a third of its width from the extrados, as will be seen later.

similar to those which had already been obtained by M. Audoy, according to Coulomb's theory; but they have added various remarks or applications which give their research a particular character of originality.

Thus, for example, starting from the hypothesis that the joint or division planes of the vault, instead of being normal to the intrados according to usage, are directed vertically and parallel to the axis, through the band of this vault and its overload, a hypothesis favourable to the solidity of the constructions and to the simplicity of the formulae or analytical expressions, they determine. By a priori considerations, relative, as always, to the average profile of a barrel vault or cylindrical vault, they determine the influence of an overload more or less close to the breaking point of the loins, its best distribution around this point, and we conclude this theorem which is subordinate to the admitted hypotheses, but which deviates very little from the truth for the lowered vaults: The point of rupture on the intrados is such that its tangent will meet the horizontal of the top of the extrados where the two half-vaults reciprocally and symmetrically butt against the vertical of the centre of gravity of the upper part, active or active, of the half-vault to which this point of rupture belongs.

The authors derive from this a graphic procedure for determining this same point, by means of an auxiliary curve which, like the geometrical trial and error indicated by Coulomb, has no other difficulty than the determination of the centres of gravity or the moments of the upper parts relative to each of the hypotheses made about the position of the point of rupture. The analysis relative to the calculation of a circular barrel vault, also extra-dossed or level, is moreover extended here to the case of spherical or domed vaults, by supposing their division into spindles, by vertical meridian planes, and the authors bring out this remark used since then for the formation of practical Tables, that "in similar vaults, the position of the rupture joint does not depend on the absolute dimensions, or is simply a function of the ratio of the radii of the intrados and extrados.

Finally, Messrs. Lamé and Clapeyron were the first to indicate the analytical way by which one could, under the same hypotheses of equilibrium by rotation, determine, in profile, the shape of the curves of the intrados or extrados of cylindrical vaults, which correspond to the maximum of stability or to an equal stability at all points; but these were only simple indications, envisaged mainly from the mathematical point of view, and to which the authors attached no practical interest. However, one cannot consider in this way the remarks with which we end their interesting Memoir, and which show the importance that there would be to establish an exact distribution of the pressures on the real joints of the voussoirs, for the large vaults in which these pressures, concentrated on a very small extent, can acquire enormous values in the vicinity of the key and the lower points of rupture, which, in the absence of any rigorous solution, requires, on the part of the builder, care and precautions of all kinds in the execution and the deformation of such vaults. Messrs. Lamé and Clapeyron did not set out, in their Memoir, the method by means of which one could approach mathematically the problem relating to the exact distribution of pressures, but it must be recognised that their rapid indications must have contributed to fixing the attention of engineers on the utility of such a solution.

As for the formulas or particular equations which they arrived at for the determination, in each case of application, of the thrust and the points of rupture of a vault, it does not seem that the hypothesis of the vertical joints brought to their resolution or numerical calculation, any notable ease, compared to those which had obtained, a little before, M. Audoy, in his *Memorandum* of 1820. The truth is that the simple and general analytical form which results from this hypothesis for the expression of the indefinite integrals of the areas and moments of the various portions of cylindrical vaults, allows us to approach with a certain ease the questions relative to the equilibrium of these vaults, However, the explicit and numerical determination of these integrals for a vault of given shape and the resolution of the transcendental equation which serves to find the rupture joint, does not require any less calculations and extremely painful trial and error. This circumstance, together with the reluctance of engineers to admit the hypothesis of vertical joints and the division of vaults which follows, however favourable it may be to stability; the necessity, finally, of having regard to the true direction of the joint planes in questions relating to sliding and pressure, explain moreover

why the method we are dealing with had not been adopted up to now by authors who have been more particularly concerned with the applications of Coulomb's theory.

The research of Messrs Audoy, Lamé and Clapeyron was soon followed by the publication, in 1825 and 1826, of the learned lessons of Messrs Navier and Persy at the École des Ponts et Chaussées and at the École d'application de Metz. The last of these professors attached himself above all to developing the theoretical ideas of Coulomb, by considering separately the various distinct modes of rupture of vaults by sliding and rotation, without, so to speak, borrowing anything from experience. He also took up, in the same spirit of abstract and mathematical discussion, the applications which M. Audoy had already made, of Coulomb's theory, to the various barrel vaults used in practice, applications to which he joined, following the example of MM. Lamé and Clapeyron, the analytical determination of the thrust in dome or spherical vaults, extradosed in parallel, followed by other applications relating to the vaults of edges and arcs of cloister, whose formulas, founded on plausible hypotheses, are, if I am not mistaken, presented here for the first time to engineers. Unfortunately, all these formulas and the calculations they require retain a frightening complication, which makes it very difficult to verify the stability conditions relating to the various hypothetical modes of rupture envisaged by the author, and the discussion of which is not always free of a certain obscurity.

In the important work already cited, M. Navier, after having exposed the various known facts of experience, relating to the stability and the rupture of the equilibrium of barrel vaults, limits himself to a very clear and very succinct summary of the general conditions of this equilibrium, with regard to the friction and the cohesion of the mortars on the joint of the kidneys, as well as the procedure to follow to ensure, a posteriori, if a given vault is stable with regard to the various modes of rupture of which it can be susceptible. But these general indications, in conformity with the principles laid down by Coulomb, are not accompanied by any of the special applications which occupied Messrs. Audoy, Lamé, Clapeyron and Persy, and they leave to the engineer the care of regulating, in each case, by trial and error, the overthickness of the uprights and the arrangement of the overloads of the vault, necessary to ensure its perfect stability; which really constitutes the principal difficulty of the question, and does not remove anything from its indeterminacy and its practical uncertainties. On the other hand, M. Navier follows these general indications with considerations relating to the elasticity of the materials of a vault supposed to be in stable equilibrium, to the mode of distribution of the pressures on the joint or support planes, and to the effects of compression or rupture by crushing which can result from it. These considerations are very delicate and all the more important because they have become, as we shall see, the starting point of a new method of direct solution of the problem concerning the stability of cylindrical vaults, a method in which Perronet's practical rule is entirely left aside, in order to calculate the thicknesses at the key, as well as those of the straight legs and the kidneys; The same is true for the other two types of objects. It is very certain, moreover, that this is not the case.

It is very certain, moreover, that in the conditions of equilibrium of an ordinary vault, and not of a small model, the resultant pressures on the joint planes cannot, according to Coulomb's own remark, be directed anywhere on the rotation edges, which they would infallibly crush; to avoid this rupture, it therefore becomes necessary to give the support surface an extent in relation to the quality of the materials. On the other hand, if there is stability, the resultants in question are completely indeterminate in size and position in the hypothesis of perfect solidity. To avoid these difficulties in the verification, a posteriori, of the conditions of equilibrium relative to a given vault, M. Navier assumes that, for the most compact joints, which we continue to call rupture joints, the normal pressure relative to the surface unit must not exceed a certain fraction of that indicated by the experiments on instantaneous crushing, in the vicinity of the supposed edge of rotation where it is strongest, nor become negative at the opposite edge where the joint virtually tends to open. It also assumes, in accordance with the hypotheses already accepted in the theory of the resistance of elastic solids, that between these two extreme points, the elementary pressures, always related to the unit of surface, increase proportionally to their distance from the first of these edges; from which it follows: 1st, that the resultant of the normal pressures in the joint must pass at a distance from the

most compressed edge equal to one third of the effective width of the joint; 2nd, that the pressure at this edge is double that which would occur if the pressure were distributed uniformly over the entire surface of the joint.

These results, in which one makes a complete abstraction of the influence of the components parallel to the planes of the joints and of the molecular displacements which result from them, displacements whose effects have not been observed at all during direct and small experiments on the crushing of materials, allow M. Navier to calculate new values of the horizontal thrust at the key, an pea stronger than those which are concluded from the strict or mathematical equilibrium relative to the hypothesis of a perfect solidity, and which offer the means of ensuring, according to the above propositions, that, in the vault in project, the materials do not run any risk of being crushed. Nevertheless, this learned engineer is careful not to conclude, from this fact and from the limit that he himself has assigned to the permanent loads to be supported by these materials, the thickness at the key or at the base of the straight legs. The state of stability or instability of a vault depends, in fact, let us repeat, and whatever one does, on that of the lower parts, on accidental or momentary overloads, as well as on any other causes of shaking or breaking of the equilibrium. If one considers, moreover, that the state of reciprocal compression of the voussoirs depends essentially on the mode of execution and the splitting of the key joints, etc., one is necessarily brought back to the precepts which derive, for each case, from the application of the calculation to existing constructions, as Mr. Audoy has done partially, it is true, by the determination of a coefficient of stability which, basically and as we have seen, also ensures that the support surface of the footings has a width which is always necessary and sufficient.

Moreover, all enlightened engineers who are afraid to leave anything to chance, regret that M. Navier did not undertake comparisons of this kind with regard to the pressures supported by the joints of vaults, subjected to the test of time, and that he did not follow the luminous exposition of the principles with a series of applications to special cases, as he has offered so many useful examples in the part of his lessons which concerns the resistance of solid elasticities. These applications, by better fixing the ideas on the value and the extent of these same principles, would at least have prevented the drawing of false consequences in practical cases.

This is also why one must attach great importance to the results of the painful studies by which his successors sought to simplify, more and more, the application of the formulas and their reductions in Tables which limit the number of calculations or trials necessary to the establishment of the vaults. Among these useful researches, we shall first rank those which appeared in 1835, in no 12 of the *Mémorial de l'Officier du Génie*, and which have mainly for starting point the Memorandum of M. Audoy.

M. de Garidel arranges one transforms the equations and analytical formulas obtained by this learned engineer, in such a way as to reduce the calculation to that of certain arguments or trigonometric functions of the angle relative to the unknown joint of rupture or maximum of thrust, functions which the author reduces to numerical Tables, which are frequently represented in this kind of questions and of which Messrs. Lamé and Clapeyron had already offered an example for the function $\alpha/\sin \alpha$. M. de Garidel gives, even for some cases, approximate formulas which allow to calculate directly, or by a very small number of numerical substitutions, the maximum of thrust and the angle known as of rupture, in the case of a vault overloaded or not with earth and masonry. But the author does not limit himself to the cases treated primitively by M. Audoy, he adds to them the case of vaults with elliptical intrados, extradosed in parallel or in screed, whose trace, the execution in large size would not offer any more serious difficulties today and which must continue to make them proscribe by 10 builders.⁵

⁵ M. de Saint-Guilhem, chief engineer of the Ponts et Chaussées, has since treated, but from a different point of view, the same question of elliptical vaults, in one of the *Memoirs* published in 1848, by the Academy of Sciences of Toulouse. In spite of the happy simplifications brought to the formulas and the methods of calculation, in spite of the numerical Tables with which they are accompanied, we doubt very much that the engineers grant to this new solution, an exclusive preference on that of which it is a question especially since the author allots to the extradados of the vaults,

M. Petit, in a Memorandum which immediately follows the previous one, also introduces into the formulas of M. Audoy, some simplifications which consist mainly in highlighting the ratio of the radii of the extrados and the intrados, which, for similar vaults, remains constant as well as the angle of the breaking joint, whereas the thrust itself, the weights and the moments of the voussoirs become proportional to the square or the cube of one of the radii. He thus manages, for vaults of this kind, round-headed and arched, extradosed in parallel, level or in a 45-degree clevis, to draw up Tables which require, so to speak, no calculation on the part of the engineers, since they contain formulas, all prepared, to calculate the thickness of the straight legs in the hypothesis of strict equilibrium and the stability coefficient 1.90, obtained by M. Audoy, for the strongest of vaults. The author has accompanied his Memorandum with useful calculations and remarks, in particular on the limit, always finite, of the thickness of the straight legs, whatever their height. The welcome given to these Tables by engineers, the numerous applications they have received in the construction of gunpowder magazine vaults and barracks whose upper storey, often very high, must be bomb-proofed by a thick overload of earth, are proof enough of the soundness and safety of the methods used to establish them.

The Memorandum of Mr. Petit is followed, in the same Collection, by a purely graphic solution of the principal questions relating to the stability of vaults, which offers the advantage of applying to any extrados or intrados forms, and of requiring, so to speak, no calculation in the discussion of the various cases of equilibrium and rupture, but simply the trace of an outline familiar to any engineer in charge of the drafting of a project. This method, which consists mainly in the graphic and linear representation of the areas and moments of a sum of segments,⁶ is accompanied by formulas and constructions relating to the case where one would like to be aware of the influence of the cohesion of mortars on the planes of joints, which one neglects, with just reasons, in the usual and practical solution of the problem of vaults. Taught for a long time at the École des Ponts et Chaussées, it continues to form the basis of the lessons at the École d'application de l'Artillerie et du Génie, where it has received simplifications which make it even more usual or faster, and which, even because of the great facility which it offers for vaults of any shape, have led Messrs. Ardant and Michon, to the establishment of more extensive Tables than those already possessed, and which were finally completed, by the latter engineer, in an important work inserted, in 1848, in if 15 of the *Mémorial de l'Officier du Génie*.

The 66 Tables in question cover almost all the cases of application relating to round arch, semi-circular and basket-handle vaults, used in practice and extradosed in variously inclined copings. They provide all the essential elements of these vaults, whose stability is ensured here by a form of extrados, thicknesses at the top, at the kidneys and at the naissances that are very suitable, and determined, as well as the coefficient of stability, according to the example of existing constructions, the lightest or the strongest, which assign to these coefficients variable values

with a view to facilitating the calculations, This new solution has an exclusive preference over the one in question, especially since the author attributes to the extrados of the vaults, in order to facilitate the calculations, a particular form which is not very economical, and that, while rejecting the coefficient of stability, under the very improper name of coefficient of correction, he substitutes for it the consideration of a surcharge at the top, the choice of which leaves a lot of room for arbitrariness, since it responds to only one cause of instability and does not rely on any previous data from experience. Let us say, moreover, that after having indicated, in the Notes of his Memoir, a new empirical formula to calculate the thickness at the top of the vaults, M. de Saint-Guilhem exposes a rather expeditious graphic method, to determine approximately the values of the areas and the moments of the voussoirs relative to a vault of any profile.

⁶ It should be noted that in those constructions where infinitely thin voussoirs are considered, the centre of gravity of the area of these voussoirs has been assumed to coincide with the middle of the corresponding thickness of the vault, whereas it is distant from it, beyond that, by a quantity $e^2/6(2r+e)$, e being this thickness, and r the radius of curvature of the extrados.

Although very small for large vaults, such a difference nevertheless has an appreciable influence on the results of no. 18, where the factor $e^2 + e^2/2r$ should be replaced by $e^2 + 2e^2/3r = e^2(1 + 2e/3r)$, which is easy to evaluate and construct in each case.

⁷ We owe to this officer, in particular, a very ingenious means of quickly constructing the linear quantity representative of the sum of the moments of the voussoir profiles, by considering the well-known properties of the funicular polygon.

between 1.5 and 2 for the straight feet; the coefficient relating to cases of sliding and rotation on the naissances, being able to be much lower, thanks to the system of construction in use.

In addition, the author of these new Tables does not conceal their insufficiency for the definitive and practical restoration of vaults; he is careful to show, by examples, how one must behave in the calculations relating to each special case, and in particular how one can have regard to the quality of the materials in order to avoid their crushing on the edge of the straight legs, where he supposes the load to be uniformly distributed over an area of the base, equal to approximately double the interval which separates this edge from the point of application of the resultant.

We have insisted a little on the work of M. Michon, whose Tables can be considered as the indispensable complement of those of M. Petit, because, without any doubt, they will exert a no less great influence on the projects to come concerning the establishment of vaults. It now remains for us to expose the attempts made to generalise the very principles of the theory which serves as a starting point for the useful work we have just mentioned.

These attempts are based mainly on the consideration of the polygon which has for vertices the centres of pressure on each of the planes of the respective joints, that is to say the points of application of the resultants of pressures or reactions on these same joints; This polygon changes into a continuous line when the joints are indefinitely multiplied, and of which Coulomb was the first, if I am not mistaken, to give a precise indication in his Memoir of 1773, by showing that, in the case where the joints are assumed to be without friction or cohesion, this curve, normal to the planes of the joints, would be, by the same token, parallel to, or rather, equidistant from, the director of the intrados of the vault.

Since then, the line of the resultant pressures has been studied by M. Moseley, in a very special and mathematical way, in a Memoir read, in June 1837, at the Cambridge Philosophical Society (volumes V and VI of the Transactions), and in various other writings published in 1839,⁸ where, under the name of line of resistance, it serves to discuss geometrically the state and conditions of equilibrium of a heavy mass composed of any number of solids in immediate contact, such as a vault with or without straight legs, buttresses, surcharges, etc. This line, which is determined by a single pressure resultant, which passes entirely inside the mass when there is stability, which intersects its free exterior surface at a certain angle when the rupture is immediate in the corresponding region, which simply touches it at the points where the rotation of the voussoirs is imminent and corresponds to the state of strict equilibrium, this line serves M. Moseley to explain the phenomena of rupture of the vaults observed by Professor Robison and M. Gauthey who had also had the opportunity to see the results of the study. Gauthey, who had also had an instinctive feeling for its existence, and he shows how the conditions of strict equilibrium to which it leads agree with those hitherto admitted according to Coulomb. Unfortunately, the search for the equation of this same line in the case of circular vaults, the manner in which the author determines the thrust at the top, and the purely mathematical discussions which follow, offer complications and uncertainties inherent not so much in the abstract point of view in which he has placed himself, as in the analytical difficulties of the problem itself. However, by limiting the formulas to those which relate to the case of ordinary rupture, they implicitly include that Messrs Audoy, Petit and de Garidel have submitted to the calculation under conditions of overloads, which are, to be sure, less general, but also more immediately applicable to the practice of engineers.

Let us note, moreover, that this learned professor did not limit himself in his research to the consideration of the conditions of equilibrium or stability which concern the rotation to which the consideration of the line of resistance is particularly applicable; but that he also considered the cases of rupture relating to the reciprocal sliding of the voussoirs, or any parts of the vault, on the joint planes, planes whose position he determines by means of a second curve which he calls the line of

⁸ A Treatise of mechanic applied to the arts; London, 1839; Theoretical and practical papers on bridges, etc. This last memoir is part of the November 1839 issue of a Treatise on Bridges, published in London by Mr. James Hann, whose own work has helped to spread in England the knowledge of French engineers on this subject.

pressures, and which is the envelope of the positions occupied by the resultants mentioned above, as the other is the place of the meeting points of these respective resultants with the joint planes which correspond to them. It is obvious, in fact, that the smallest angle at which the first of these lines meets these various planes determines the one among them where sliding becomes virtually possible and inevitably takes place when the angle in question is less than the complement of the friction of the surfaces in contact. Here again, the author has remained in generalities which are valuable, no doubt, from the geometrical point of view, but whose difficulty and uncertainties in application are due to the very indeterminacy of the curves of resistance and pressure.

We will apply similar reflections to the content of an interesting Memoir by M. Méry, on the equilibrium of barrel vaults, dated February 1839, but which was not published until 1840, in the *Annales des Ponts et Chaussées*, and in which he deals more specifically with the geometric and mechanical properties of the first of these lines, called here, in reverse, the pressure curve. Indeed, the construction and the equation that the author gives for this curve, offer an arbitrariness that can only be avoided or removed by means of the very data provided by experience or numerical calculations relating to the position of the rupture joints in each special case. Nevertheless, the large number of examples of vaults, shown in the plates which accompany Mr. Méry's Memorandum, and where the pressure curve is approximately traced according to the results observed or calculated previously by Messrs. Boistard and Audoy, give, of this curve and of the position of the joints of rupture in the vault, the buttresses or straight feet, an intuitive feeling which also has its purpose of utility, and which did not fail to exert its share of influence in the practical or theoretical solution of the problems concerning the stability of the vaults.

This observation is applicable, especially, to the conditions by which this young engineer seeks to regulate, a priori, the thickness that a vault would claim at the various points, because of the resistance of the materials and the overloads to which it would be exposed. Starting from principles similar to those used by M. Navier, on the compressibility of elastic solids and on the distribution of pressures along the planes of joints exposed to open internally or externally, he arrives at consequences which are perhaps debatable in all rigour,⁹ but which, basically, differ little from those of this illustrious engineer, for the fixing of the minimum interval to be observed between the curve of pressures and the lines of intrados or extrados of the vault. This interval being once set, by taking it, for example, as M. Navier wants, a third of the length of the vault. Navier, a third of the corresponding length of the joint, makes it possible to draw two curves respectively equidistant from these lines and in the interval of which must be found the curve of the pressures relative to the state of stability of the system; but this curve itself remains undetermined, unless one fictitiously supposes the rupture of the equilibrium by rotation around the edges of the planes of the joints limiting to the new intrados and extrados, which would require calculations or trial and error that are very painful and not very justifiable in principle.

⁹ M. Belanger, in his lessons at the *École des Ponts et Chaussées*, adopting, in part, the ideas of M. Méry, rectified them by generalising the hypothesis of M. Navier, relating to the distribution of pressures between two limit elastic solids at a common joint plane. Navier, relating to the distribution of pressures between two elastic solids limited to a common joint plane, a hypothesis which amounts, in reality, to supposing the elementary pressures at each point to be proportional to the ordinates of a plane infinitely close to the first, and for which they measure, as it were, the corresponding molecular and relative displacements, supposedly parallel to the general resultant. Only here this plane is directed in some way, although parallel, with respect to the most compressed edge, and the point for which the pressure would be zero is no longer necessarily situated at the opposite end of the joint, as M. Navier has assumed. In a word, M. Belanger replaces the consideration of the compression triangle by that of any trapezium determined by the knowledge of the resultant.

We have said nothing of the analytical considerations which conclude Mr. Méry's Memoir, because they do not bring any new or essential element to the practical solution of the problem of vaults, and they even contain some doubtful assertions concerning the nature of the curve of pressures and that of the centres of gravity, the coincidence of which, according to the author, would inevitably lead to the identity with the ordinary funicular or chain, in vaults of constant thickness.

It is a long time since Coulomb, in his famous Memoir of 1773, and, after him, Salimbeni, an officer of the Italian Engineers, in a work published in Verona, in 1787, showed, in truth for particular cases, that if this identity took place in the hypothesis of an infinitely thin vault, it was no longer the case in that where the thickness was finite, The differential equation of the intrados curve remains a function of this thickness and retains a more general form than that of the ordinary chain, even for a vault of constant thickness, a case for which the integration is reduced, by the second of these authors, to a simple quadrature belonging to a higher class of transcendents.¹⁰

Salimbeni has exclusively considered vaults in the ancient hypothesis where, neglecting friction and cohesion on the joint planes, each of the voussoirs must be in equilibrium separately under the action of its own weight and the resultants of pressures and reactions coming from the neighbouring voussoirs; It also assumes that these resultants, perpendicular to the respective joints, are naturally directed, on either side, towards the centre of gravity of the interposed segment; this makes the line of pressures coincide with that of the centres of gravity, to which the joints are thus supposed to be normal, at each point. Seeking, under these particular conditions, the analytical expression of the infinitely small difference of the pressures exerted on either side of a same joint, and which come from the weights of the contiguous segments, decomposed perpendicularly to this joint and its opposites; By equating this difference to zero in order to express the mathematical conditions of equilibrium in the polygon of the centres of gravity, the author arrives, by a very laborious route, at the above-mentioned differential equation and its indefinite integral, without, moreover, bothering in any way to determine the arbitrary constants according to the physical or practical data of the problem.

Salimbeni's work, which is no less than three hundred pages long, contains a large number of propositions concerning the equilibrium of barrel or dome vaults, of a shape given a priori, and where, consequently, the above conditions of stability are not rigorously satisfied, even in the case of bands without overloads, of uniform thickness, etc., of which the semicircular vault offers a particular example. Also, the results he arrives at for the determination of the thickness of the straight legs are marred by the same vice, the same exaggeration, as those obtained by Lahire and Belidor, although, from the theoretical point of view, they perhaps offer less uncertainty and arbitrariness.

Salimbeni also examines the case of overloads limited by an upper profile of some continuous shape, which he usually assumes to be subdivided into infinitely thin vertical prisms; but, instead of admitting, with most engineers, that the weight of these prisms is simply added to that of the corresponding vault, an assumption which is very favourable to solidity, since it takes no account of the mode of connection of these prisms, he is led to consider this weight to be decomposable into two forces: one horizontal, perpendicular to their mutual support surface and which must be destroyed by the resistance on reaction of the right foot and of the external mass and band of the half-vault; the second perpendicular to the direction of the corresponding element of the extrados against which the reaction in question is supposed to retain the superimposed prism, without friction or cohesion, as if on a small inclined plane, thus pressing it there with a normal effort, obviously equal to that which would take place in the hypothesis of the liquidity of this same prism. The normality of this effort becomes, in fact, indispensable to ensure mathematically the individual equilibrium of the voussoirs of a vault extradosed in parallel and where friction and cohesion on the joint planes are neglected, since only then can the forces be supposed to concur at the centre of

¹⁰Degli archi e delle volte, lib. V prop. 8, carol. 3.

gravity of each of these corners or elementary voussoirs; any tendency to rotation being, by the same token, prevented or destroyed.

On the other hand, if we break down the normal pressure mentioned above into its vertical and horizontal components, the first will obviously reproduce the weight of the vertical prism superimposed and which corresponds to the ordinary hypothesis, the second will give, in addition, a force equal to that which holds this prism, without friction, against the extrados, and which, acting from the outside to the inside of the vault, will increase the thrust or reaction at the top and the moments relative to the rotation around the lower support points. The hypothesis admitted by Salimbeni on the influence of the upper overloads, therefore, seems even more favourable to the stability or solidity of the system than the one in question; but it must also lead, in the determination of the lateral and lower support masses, to exaggerations of thickness which would need, as always, to be justified a posteriori, by the comparison of the results of the calculation with the data of experience offered by the already existing constructions; all the more so because, whatever the precautions used in the execution and deconstruction of a vault, one cannot prevent the movements and effects of settlements which modify to a greater or lesser extent the state of stability or reciprocal compression of the voussoirs, assumed by the calculations.

Finally, the difficulty one experiences in following the ideas and demonstrations of the Italian author, all synthetic and in the manner of the ancients, notably Huygens, would have turned us away from mentioning his work, if, even today, the subject he deals with did not offer a certain interest from a mathematical point of view and the happy attempts that have just been made by M. Yvon Villarceau to study the structure of the vaults. Yvon Villarceau, to determine the most advantageous form or maximum stability of the vaults, that is to say, disregarding all considerations of friction and cohesion on the joint planes.

In a first Memorandum on cylindrical barrel vaults (*Revue de l'architecture et des travaux publics*, by M. Daly, volume V, 1845), M. Yvon supposes that, in the profit, the line of the centres of gravity is loaded, at each point, with the corresponding elementary voussoir, and that the joints feel everywhere normal to this curve that the resultants of pressure or tension acting perpendicularly to the whole surface of these joints, pass by the centre of gravity of the voussoir, and that, in addition, if there are external or foreign forces to the vault band, they are also directed towards this centre, which necessarily reduces to two the equilibrium equations expressing that the sum of the vertical forces and that of the horizontal forces are separately zero; The problem here differs from that of the catenary only in that the weights and tensions acting at each point are implicit functions of the density, shape and dimensions of the segment itself. After expressing these forces as a function of the width of the joint plane or corresponding thickness of the vault, the average pressure per unit area on this joint, the radius of curvature and the interval between the centre of gravity and the middle, the thickness, M. Yvon integrates, in the first place, the two equations thus transformed, in the case where one would suppose the external forces to be null and the average pressure or tension to be constant and equal to the limit of that which the materials of the vault can support, in a permanent way, according to experience, this limit being expressed here in height of a column of the same material and weight, this makes him know the law, very simple the increasing thicknesses of the vault from the top where it remains entirely arbitrary and must be, consequently, determined by the sole use of the builders, until the birth of this supposed to rest on an unshakeable cushion and whose inclination is fixed by the very data of the calculation, as well as the line of the centres of gravity, expressed by a transcendental equation of a very simple form, and which makes it possible to quickly draw the curves of intrados and extrados of the vault.

The results of this analysis are then applied, by the author, to the case of an unloaded vault which must simply cover a space of assigned width, as well as to the case where the vault must support, towards the top, a surcharge formed by the foot-droît or the oblique fall of another vault given a priori; The last of these problems requires that the dimensions and shape of the keystone used as a bearing be determined beforehand; this is done by considerations and calculations that are both simple and ingenious, but which, unfortunately, are not destined to receive many practical

applications. Passing finally to the vaults or arches of bridges, in which the band is surmounted, usually, by a tympanum or surcharge of masonry, earth, etc., limited to an upper horizontal plane, M. Yvon remarks that the pressure exerted by such a surcharge on the extrados of the vault may vary with the nature of the construction adopted, and offers a real indeterminacy, since its value, which would be almost nil in the case of a system of ashlar forming above this vault a real arch-buttress by the very dimensions or mode of superposition of these stones,¹¹ could, in other cases, acquire an intensity comparable to that of a liquid of a density equal to the average density of the surcharge.

This reason and those already explained in connection with the work of Salimbeni, lead the author to definitively adopt the hypothesis of liquidity, where the pressures normal to the curve of the centres of gravity, directed towards the interior or exterior of the curvature, as in underground water pipes, are proportional to the extent of each of its elements, as well as to the corresponding vertical thickness of the surcharge; This leads, as another forced consequence of the individual equilibrium of the voussoirs without friction, to the parallelism, the almost equidistant curves of the intrados and extrados in relation to that of the centres of gravity. Also, to avoid the contradiction which would take place between this assumption and the results of the calculation in the case where the nature of the overload would require a progressive repaïssissement of the band of the vault starting from the top, the author is led to consider a mode of construction of the extrados with redans, purely fictitious and suitable only to justify the assumptions of the calculation, where one makes completely abstraction of any friction of the surfaces in contact. We shall soon see the author abandoning this arrangement, which would be difficult to adopt by engineers, and returning to the use of a continuous upper surface parallel to the line of the centres of gravity and whose normal pressures therefore pass naturally through these respective centres.

The integration of the equations relating to this case, by means of methods of approximation or of the trace of the curve by its radii of curvature, the determination of the arbitrary constants, either in the hypothesis where the weight of the vault could be considered as entirely negligible with respect to that of the overload, or in that which would be very comparable to it numerically, and which relates to the vaults of bridges overloaded with a tympanum limited to an upper horizontal plane, this integration, this determination, This integration, this determination, I say, as well as the research of the thrusts against the straight legs or massive abutments lead the author to very complicated results and methods of resolution, always ingenious from the mathematical point of view, but which, in this state, could hardly be useful to practising engineers, since, resting on the hypothesis where the average pressure remains constant in each joint and where the thickness of the vault grows from the top to the naissances, one is led to the difficulties already mentioned above. Also, M. Yvon Villarceau has courageously taken up the question relating to the arches of straight bridges, in a Memoir which, presented in November 1845, to the Academy of Sciences, was, the following year, the subject of a very favourable Report, according to which the Academy ordered its printing in the *Recueil des Savants étrangers*: a In summary, says the learned rapporteur, M. Lamé, the work of M. Yvon Villarceau is remarkable. Yvon Villarceau is remarkable on more than one point; in addition to the new views which concern the theory of vaults, it offers a curious example of the utility of elliptical transcendents; the calculations and especially the methods of approximation are

¹¹The examples of such constructions, which are basically related to the class of corbels, systems with crossettes, discharges, etc., are not lacking in ancient buildings, where they take the place of the more modern voussoir systems. The examples of such constructions, which are basically related to the class of corbels, systems with crossbars, discharges, etc., are not lacking in ancient buildings, where they take the place of the more modern systems with voussoirs, and suppress in some way, by the large size of the stones, any horizontal thrust, to the detriment of true solidity, thus bringing into play in an often very energetic manner, the oblique or relative resistance of the materials, the source of so many landmines in the public monuments of Egypt, Greece, and France. Rondelet, in his *Art de bâtir*, states moreover as a fact of experience, that five or six courses of ashlar, superimposed on a semicircular vault, are enough, not only to load it with the weight of the upper constructions, but also to completely cancel its horizontal thrust. The example of the mines just mentioned does not allow such a principle to be used for such constructional devices, and one should not be surprised to see the engineers of the public services take no account of the influence which may be due to the relative or transverse resistance of the stones and the cohesion of the mortars on the joints of a vault.

handled there with an unusual dexterity, "(Comptes rendus, tome XXIII, p. 866 and following).

It will suffice to recall that the author, abandoning here the arbitrary condition of the uniformity of the pressures in the whole extent of the vault, is led to give to the band or rather to the surfaces of reciprocal support of the voussoirs, a constant width limited by an extrados and an intrados parallel to the line of the centres of gravity, and which are equidistant to it, but of which the last one is only purely fictitious, and must be replaced, at the time of execution, by a real intrados, brought closer to the interior of the vault by an always very small quantity, and which increases approximately in inverse reason of the radius of curvature in each point, in order to take account of the analogous interval which must necessarily exist between the average line or middle of the planes of joints and that of the centres of gravitation. In this way, in fact, and by means of a compression of the joints equal in depth to the interval which separates the real intrados from the fictitious intrados, M. Yvon succeeds in satisfying all the mathematical conditions of the problem, since the resultants of the pressures passing through the middles of the bearing surfaces on the joint planes, there is every reason to admit that the elementary and reciprocal pressures will be found, by the same token, uniformly distributed over this surface. Nevertheless, there is such a slight difference in the position of the mean line in relation to the centre of gravity, for large parallel extradosed vaults, that there would be no disadvantage in neglecting it in the calculations as well as in the execution.

Let us add that the extrados of the vault being here continuous and parallel to the line of the centres of gravity, the author, in order to carry out the condition relating to the normality of the external pressures, is satisfied to suppose that, in the execution, the prismatic stones which surmount the surface of the extrados, will be placed upright and finished with a whistle cut, which allows them to slide along the elements of this surface, assuming that their mutual friction is neglected as Salimbeni did. But, in addition to the fact that this kind of cut may result in difficulties of execution which are repugnant to engineers, it is also doubtful that they would consent to lay stones which must bear very heavy loads towards the lower parts of the vault, where such combinations would at most be applicable to the two heads. However, this consideration, which will maintain the stepped construction in the establishment of the vaults of large bridges, is by no means a reason to reject, a priori, the deductions of the author, based on the extreme supposition of the perfect liquidity of the overloads, very favourable, as we have seen, to stability, especially if the resulting dimensions for the abutments, do not offer anything exaggerated from the economic point of view.

In order to accomplish the task we have set ourselves, it remains for us to mention the recent and thorough studies, perhaps too extensive, that Mr. Scheffler, of Brunswick, has undertaken on the geometric and mechanical properties of the pressure curve, which this engineer mistakenly attributes to Méry, who, as we have seen, did not take up the subject until a few years after Moseley. These new researches, which appeared in German, in volume XXIX (3rd book) of the *Journal des Constructions*, by M. Crelle, concern rather the point of view. It is a question, in a word, of generalizing, by rectifying and completing in some points, the principles which served as a basis for the existing theories, by considering that, in the state of equilibrium of a vault, the resultant of the pressures on the vertical joint of the key, must satisfy the condition of the minimum according to a principle put forward by M. Moseley (*Philosophical Magazine*, p. 4). Moseley (*Philosophical Magazine*, October 1833), under the name of principle of least resistance, and of which M. Scheffler has then extended and brought to light, the true meaning in a previous Memoir inserted in volume XXVIII of the already quoted *Journal* of M. Crelle.

This principle, whose statement, obvious enough in itself, could be generalized further, does not differ, in substance, from that of which Coulomb offered, in his Memoir of 1783, such beautiful applications to the theories of the resistance of solids, of the thrust of earth and of vaults; The whole difficulty being only to draw from it, without too much trial and error or calculation, exact consequences, relative to the form and position of the line of pressures, apart from the case where it is supposed to have to pass through certain given points, a priori, as MM. Moseley and Méry, by placing themselves frankly in the conditions of strict equilibrium, envisaged, by Coulomb, as simple limits to the stability of vaults. Now this is precisely what Mr. Scheffler proposes to do, while

remaining in the hypothesis of the perfect incompressibility of the voussoirs, and without borrowing anything from experience or from the data previously acquired on the different modes of vault failure.

In spite of the generality and the complexity of the point of view on which the author of the new studies on the properties of the line of pressures has placed himself, and although at first sight, it seems to attribute to this line a form which cannot exist in the physical and practical conditions of the vaults, the discussions which emerge from it will not less contribute to illuminate more and more the bottom of the theoretical question, which had perhaps obscured some of the commentators of Coulomb and M. Moseley, quoted by M. M., and which is the subject of this paper. Moseley, quoted by Mr. Scheffler, who moreover indicated, in his Memoir, a graphic procedure, in itself quite simple and founded on the hypothesis of the division of the vault into vertical joints, to determine approximately and under certain data, the curve of the resultants of pressures, belonging to a portion of cylindrical vault of an assigned profile and which satisfies, or not, the condition of ordinary symmetry of its parts or of its overloads, with respect to the vertical joint of the key. But this entirely geometrical determination, in our opinion the main point of the Memoir, requiring, so to speak, an unlimited number of constructions and trials, makes the method almost illusory in its practical applications.

M. Scheffler's memorandum will have the advantage of highlighting the work of some German or English engineers, hitherto little known in France, and which, for this reason, it would have been impossible for us to mention. Such are, in particular, the theories of M. Eytelwein and his successor, M. Unger, in which the consideration of friction on the joint planes is brought into play, albeit incompletely, in the solution of Lahire on du coin; that of Messrs. Gerstner, Knochenhauser and Schubert, all based on the existence of a certain line known as the voussoir support line, which is supposed to merge with the funicular or the chain, but which is, in reality, only the line of the pressure centres. Such are also the researches of M. Hagen, published in the Memoirs of the Academy of Sciences of Berlin, and where supposing arbitrarily that this line passes, at the same time, by the middle of the joint of the key and by that of the lower joint, one hopes to obtain a line of pressures which makes the vault enjoy the maximum stability. Such is, finally, the theory of Mr. Barlow, published in 1847, in the work entitled: *Mechanical Principals*, and in the *English Journal of Civil Engineers and Architects*, a theory in which, while adopting the ideas of Dr. Moseley, the author would also have made the mistake of confusing the line of pressure with the funicular and the chain.

In addition, this confusion of ideas noticed by M. Scheffler and which we ourselves mentioned on the occasion of M. Mery's Memoir, is found in several writings published subsequently in France, among which we will suffice to quote one which appeared in 1845 under the title of *Routine for the establishment of vaults*, and whose mode of solutions refers exclusively to the hypothesis of the sliding of the joints adopted by the former authors, a mode which we had also seen reproduced in an Italian Memoir by Dr. Vincenzo Amici (*Modène de la construction des voûtes*). This mode was also reproduced in an Italian memoir by Dr. Vincenzo Amici (*Modena*, 1833), in a form and with analytical developments which belong much more to abstract mathematics than to the practical applications of mechanics.

Such is, if I am not mistaken, the more or less exact, if not complete, summary of the efforts which before the last works of Messrs Yvon Villarceau and J. Carvallo, of which we have to give an account to the Academy, have been made in various countries to solve the so interesting and so difficult problem of the stability of vaults; a problem which, judging from the state of imperfection in which the theories relating to the elasticity of solids are still found, will perhaps wait a long time for a satisfactory solution from the physical and mathematical point of view. It is clear, in fact, from the foregoing, that the two questions of the equilibrium of vaults and the elastic resistance of solids are linked together in the most intimate way, whenever one wishes to leave the abstract hypothesis in which one assumes continuity and absolute invariability of form for the voussoirs. The analogy is even such that one can say, without going too far, that the theory of vaults and that of naturally

curved elastic solids are really only one, considered under different conditions and aspects.

There is more: the curve of the resultants of pressures, considered in the current theory of vaults, where one neglects all the molecular forces of traction which could result from the cohesion of mortars on the joint planes, this curve is represented, in a way, in the ordinary theory of the resistance of solids, by the line called, improperly perhaps for some cases, neutral axis, axis of the invariable fibres, and whose coincidence with that of the centres of gravity, is only subordinated to certain hypotheses made on the role of the molecular forces of extension and compression, in each of the transverse sections of the solid, or, more especially still, on the rigorous equality which would subsist, according to certain data of experience, between these two kinds of forces, for relative elongations and shortenings, equal and opposite, of the elementary fibres. Similarly, what is called the breaking point or joint in skates corresponds precisely to the points or sections that could be called dangerous in elastic solids, since the molecules there experience relative and maximum displacements that it would not be appropriate, according to the example of existing constructions, to subject them to a permanent manner.

Finally, it is no less obvious that any theory of elasticity, any indication of experience which would make explicit the law of deformations and molecular actions relative to an infinitely thin slice of a solid subjected to domestic compression and extension forces, would, by the same token, be very likely to lead to the analytical solution of both problems, which it would tend to bring back to those which depend on the calculation of ordinary differentials. Unfortunately, the law in question seems just as difficult to discover a priori in an infinitely thin prism as in a prism of finite size, since it depends, whatever one does, on the very equations which express the boundary conditions of the body or the law of distribution of the molecular forces on its free or external surface. It is well known that the Academy of Sciences has, up to now, vainly proposed one of its great prizes for the analytical solution of this difficult and important question, which it would perhaps be advisable to limit, for the time being, to the case of an elastic prism subjected only to the action of gravity and resting on a horizontal material plane, supposedly non-reformable, by a base whose inter-molecular distances would themselves be supposed to be invariable. We must therefore call, with all our wishes, for the improvement and development of the methods by which Messrs Navier, Poisson, Cauchy, Lamé and Clapeyron have approached, from the mathematical point of view, the questions which concern the internal equilibrium of elastic solids, and which, while awaiting an entirely rigorous solution, have already led M. de Saint-Venant to perfect the old theory of the resistance of bodies to bending, parallel sliding and transverse torsion, produced by externally and variously applied forces. The long interval which had elapsed since the first researches of the geometers first mentioned, had made the enlightened friends of science fear that this important branch of mathematical physics would fall into complete abandonment, when the remarkable work of M. Lamé¹² came to dissipate this apprehension, by giving us the hope of seeing soon the theory of vaults, itself, subjected to principles even more satisfactory than those we possess. At least, we must be infinitely grateful to this scholar for having spread over such a difficult subject a clarity of exposition which makes it, so to speak, elementary, and for having sown new and fruitful insights, which will not be long in receiving useful applications to the various problems which interest the art of engineering. Let us add that the beautiful and delicate experiments of Mr. Wertheim, on the compression of elastic solids, are another reason to hope for a prompt and happy solution.

¹² Leçons sur la Théorie mathématique de l'Élasticité des corps solides; Paris, 1852, imprimerie de Bachelier.